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ON THE BILINEAR REGULATOR PROBLEM  
WITH A PURSUIT-EVASION APPLICATION\*

by

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20. Abstract (Continued)

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## I. Introduction

The linear quadratic regulator problem has been studied intensively by modern control engineers and many fruitful results have been obtained due to its simple structure. The existence and uniqueness of an optimal control to this linear regulator problem is well known, [1, 2]. The optimal control of nonlinear systems was studied by Lee and Markus [2] and Athans and Falb [3] in which the existence and uniqueness of an optimal control to the quadratic cost problem were obtained for some nonlinear systems with a control constraint. Their results have applications to certain classes of problems including the time optimal control of a class of bilinear systems with control constrained to a compact set as reported in [4] and [5]. But, in general, the conditions are not known for the existence and uniqueness of an optimal control to the quadratic cost problem without control constraints for bilinear systems.

Bilinear systems have received attention in recent years due in part to their applications in various engineering, biological, as well as socio-economics systems, Mohler [6], and in part to their intrinsically nearly linear structure. In fact, the bilinear systems not only can represent those physical systems containing a bilinear mode, but also have potential applications in the modeling of systems containing sinusoidal nonlinearities, especially those arising in spatial flight systems. This observation motivated the study of the bilinear regulator problem discussed in this paper. The existence of optimal controls to this problem is established in Section II. The approach used here is similar to that of Lee [2]

for linear systems but extends it in conjunction with the concept of attainable set for bilinear systems.

A class of bilinear systems in which the system matrices commute with each other was studied by Sussmann [7]. It was shown that the attainable set is closed relative to "bang-bang" input functions over a given interval  $[0, T]$ . Here, the minimum energy control of such commutative bilinear systems is investigated in Section III. Sufficient conditions for the uniqueness of solutions to the two-point boundary value problem stemming from the Pontryagin Maximum Principle will be derived. This result assures that solving a nonlinear algebraic equation will lead to the optimal solution to this problem. That is, the optimization problem is reduced from an infinite dimensional to a finite dimensional problem. Furthermore, it is shown that the optimal controls for the minimum energy problem associated with the commutative bilinear system appear in a very simple form, that is, a constant vector determined by the boundary conditions. The reachability property for this class of bilinear systems is also discussed. We will show that for a commutative bilinear system, a terminal state  $x_1$  is constant reachable if and only if it is reachable by a time-dependent control. This fact provides a great deal of insight in implementing a feasible controller for this kind of system from a design point of view. It will also be noted that for this class of bilinear systems, the reachable zone is much easier to characterize because the bilinear system becomes a linear time invariant parameterized system when a constant input  $u_c$  is imposed as a parameter vector. One need only compute the corresponding

transition matrices  $\Phi(t, t_0; u_{c_i})$ ,  $i = 1, \dots, m$ , which characterize the reachable zone of the bilinear system, independent of the initial conditions.

Application to a two-dimensional pursuit-evasion problem is considered in Section IV. This problem has been studied for many years by using different formulations and control schemes. Among these, Deyst [8] applied the theory of optimal stochastic control to derive a sophisticated nonlinear control which lead to a significant improvement in performance in relation to those resulting from the conventional proportional navigation guidance law. Meanwhile, the theory of differential games was considered by Ho [9] and Rajan [10] to determine optimal strategies for this problem. Slator [11] took the perturbation point of view in applying optimal control theory to this problem. A direct approach using a bilinear regulator formulation is considered here which takes advantage of the merits of a commutative bilinear system as mentioned in Section III to analyze the nonlinear problem directly. A closed form solution to this problem is obtained. It has a great deal of potential to be implemented in practice due to the structural constancy of the controller.

## II. Bilinear Regulator Problem

Consider the single-input bilinear system<sup>†</sup>

$$\dot{x} = [A + Bu(t)]x, \quad x(t_0) = x_0 \in \mathbb{R}^n \quad (1)$$

$$t \in [t_0, T]$$

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<sup>†</sup>Notice that a direct input term  $Cu$  can always be absorbed into the bilinear term  $Bux$  by introducing an additional state,  $x_{n+1} = 1$ , to obtain this expression.

where  $A$  and  $B$  are  $n \times n$  constant matrices;  $u(t)$  is a square integrable function defined on  $[t_0, T]$ ;  $x_0$  is the initial state.

We shall first study some basic properties of the single-input bilinear system associated with a minimum energy cost, then generalize this to the multi-input case together with the general quadratic cost.

The problem is to find an input function  $u^* \in L^2([t_0, T], \mathbb{R})$ , the class of square integrable functions defined on  $[t_0, T]$ , which minimizes the following cost function

$$J(u) = x'(T)Qx(T) + \int_{t_0}^T u^2(t)dt \quad (2)$$

where  $Q$  is a  $n \times n$  nonnegative definite symmetric constant matrix and prime denotes the matrix transpose operation.

Define the set of attainability  $K$  of (1) and (2) by

$$\begin{aligned} K = \{(x^0(u), x(T; u)) \in \mathbb{R}^{n+1} : & x^0(u) \equiv \int_{t_0}^T u^2(t)dt, \\ & x(T; u) \equiv \Phi(T, t_0; u)x_0, \\ & u \in L^2([t_0, T], \mathbb{R})\} \end{aligned} \quad (3)$$

where  $\Phi(t, t_0; u)$  is the state transition matrix of (1) for each given  $u$ . It is clear that  $K$  consists of the pairs of cost and terminal state corresponding to all admissible input functions as coordinates.

Now consider the map  $G: K \rightarrow [0, \infty)$  defined by

$$G(x^0(u), x(T; u)) \equiv x'(T; u) Q x(T; u) + x^0(u). \quad (4)$$

Hence,

$$J(u) = G(x^0(u), x(T; u)). \quad (5)$$

The following lemma reveals some characteristics of the set of attainability  $K$ .

Lemma 1:

Every subset  $N(p) = \{(x^0(u), x(T; u)) \in K : 0 \leq x^0(u) \leq p, p < \infty\}$  of  $K$  is compact.

Proof:

The proof is straightforward by the Euclidean topology and the linear structure of (1) for each given  $u$ . Since  $N(p) \subset K \subset \mathbb{R}^{n+1}$ , and  $0 \leq x^0(u) \leq p$ , one can easily establish the closedness and boundedness of the set

$E_0 = \{x(T; u) : u \in L^2([t_0, T], \mathbb{R}), \int_{t_0}^T u^2(t) dt \leq p\}$ , which proves the lemma.

Therefore, for any nested sequence  $N(n) \subset N(n+1) \subset \dots$ ,  $K = \bigcup_{n=1}^{\infty} N(n)$ . The compactness of  $N(p)$  for each given positive  $p$  assures the existence of an optimal solution to the system (1) and (2) as given in the next theorem.

Theorem 1: (Existence)

Given the system (1), there exists an optimal input function  $u^* \in L^2([t_0, T], \mathbb{R})$  which minimizes the cost function (2).

Proof:

By definition of  $G$ ,

$$G(x^0(u), x(T; u)) \geq 0$$

and

$$\lim_{x^0(u) \rightarrow \infty} G(x^0(u), x(T; u)) = \infty \quad \text{uniformly on } K.$$

Hence, for every  $M > 0$ , there exists a  $p_0$ ,  $0 < p_0 < \infty$ , such that

$$G(x^0(u), x(T; u)) > M \quad \text{for all } x^0(u) > p_0. \quad (6)$$

We can always choose a proper<sup>†</sup>  $M$  such that

$0 \leq \inf_K G(x^0(u), x(T; u)) < M$ . Therefore, there always exists  $(x^0(u_1), x(T; u_1)) \in K$  such that  $G(x^0(u_1), x(T; u_1)) < M$  and  $x^0(u_1) \leq p_0$ , ( $u_1(t) = 0$  is an example). Hence, the set  $N(p_0)$  defined by  $N(p_0) = \{(x^0(u), x(T; u)) \in K: 0 \leq x^0(u) \leq p_0, u \in L^2([t_0, T], R)\}$  is nonempty. Furthermore, by Lemma 1,  $N(p_0)$  is compact in  $R^{n+1}$ . The continuity of  $G$  on  $N(p_0)$  can be easily established which assures that  $G$  attains its minimum in  $N(p_0)$ . This will imply the existence of an optimal input  $u^*$  which minimizes the cost function (2) because outside  $N(p_0)$  the cost is always greater than  $M$  as shown in (6).

Q.E.D.

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<sup>†</sup>For instance, choose  $M > [\Phi_A(T, t_0)x_0]^\top Q\Phi_A(T, t_0)x_0$ , where  $\Phi_A$  is the transition matrix associated with  $A$ ; then  $\inf_K G(x^0(u), x(T; u)) \leq [\Phi_A(T, t_0)x_0]^\top Q\Phi_A(T, t_0)x_0 < M$  where the intermediate term is the cost when  $u(t) = 0$ .

In general, for the multi-input bilinear system

$$\dot{x} = (A + \sum_{i=1}^m B_i u_i) x \quad (7)$$

$$u(t) = (u_1, \dots, u_m)'$$

where  $A, B_i$  ( $i=1, \dots, m$ ) are  $n \times n$  constant matrices;  $u \in L^2([t_0, T], R^m)$ , the class of  $m$ -vector valued square integrable functions on  $[t_0, T]$ , and the associated quadratic cost

$$J_2(u) = x'(T)Qx(T) + \int_{t_0}^T [x'(t)W(t)x(t) + u'(t)R(t)u(t)]dt \quad (8)$$

where  $Q$  is as defined in (2);  $W(t)$  is a  $n \times n$  symmetric matrix-valued function continuous in  $t$  and nonnegative definite for each  $t \in [t_0, T]$ ;  $R(t)$  is a  $m \times m$  symmetric matrix-valued function continuous in  $t$  and positive definite for each  $t \in [t_0, T]$ , a similar existence result can be derived. We state this in the next theorem.

Theorem 2:

There exists an optimal input function  $u^* \in L^2([t_0, T], R^m)$  which minimizes the cost given by (8) subject to the constraint (7).

Proof:

By taking

$$M > [\phi_A(T, t_0)x_0]^\top Q \phi_A(T, t_0)x_0 + \int_{t_0}^T [x_0^\top \phi_A^\top(t, t_0)W(t)\phi_A(t, t_0)x_0]dt$$

we can construct the set  $N(p_0)$  similar to that in Theorem 1. The closedness of  $N(p_0)$  follows from the same construction of a convergent subsequence  $u_k$  (weakly) and  $x(t, u_m)$ . The boundedness of  $N(p_0)$  and continuity of  $G$  on  $N(p_0)$  can be shown by a similar argument as in Theorem 1.

The existence results derived here allow us to apply the Maximum Principle in obtaining a two-point boundary value problem and investigate its solution either by analytic methods or numerical schemes. The minimum energy control of a class of bilinear systems in which the system matrices  $A$  and  $B_i$  commute with each other will be discussed in the next section.

### III. Minimum Energy Control of Commutative Bilinear Systems

#### Definition 1:

The system (7) is called a commutative bilinear system if every pair of the matrices  $(A, B_1, B_2, \dots, B_m)$  commute with each other.

The commutative bilinear system has been studied by Sussmann [7] in which it is shown that the attainable set of this system with bang-bang controls is closed. Baras[12] recently extended this result to the delayed commutative bilinear system. The problem considered here is the minimum energy control of a commutative bilinear system without control constraint. It will be shown that this problem has a simple solution which possesses an easily-implemented character.

Two kinds of cost functions are investigated here,

$$J_1(u) = x'(T)Qx(T) + \int_{t_0}^T u'(t)Ru(t)dt \quad (9)$$

without a terminal constraint on the state, and

$$J_2(u) = \int_{t_0}^T u'(t)Ru(t)dt \text{ with } x(T) = x_1 \quad (10)$$

where  $x_1$  is a prespecified vector.

Obviously, the problem associated with the cost (10) involves the attainability of the given system to the desired target set.

#### A. Without Terminal Constraint

Lemma 2: (Pontryagin)

The optimal solution, if it exists, of the system (7) with cost (9) satisfies the following two-point boundary value problem

$$\begin{aligned} \dot{x}^* &= (A + \sum_{i=1}^m B_i u_i^*) x^* & x(t_0) &= x_0 \\ \dot{p}^* &= - (A' + \sum_{i=1}^m B_i' u_i^*) p^* & p(T) &= -Qx(T) \end{aligned} \quad (11)$$

where

$$u^*(t) = \frac{1}{2} R^{-1} \begin{pmatrix} x^{*'} & B_1' \\ \vdots & \vdots \\ x^{*'} & B_m' \end{pmatrix} p^* . \quad (12)$$

Proof:

This is a special case of the Maximum Principle applied to the

bilinear system with minimum energy cost. The proof can be obtained in many books on optimal control theory, see Pontryagin [13].

The commutative structure of a bilinear system allows us to establish the analytic property of the optimal controls for the above problem. This is illustrated in the following theorem.

Theorem 3.

Given a commutative bilinear system (7) with the associated cost (9), then the optimal controls which minimize the cost (9) are in the form of a constant vector which satisfies the algebraic equation:

$$u^* = \begin{pmatrix} u_1^* \\ \vdots \\ u_m^* \end{pmatrix} = -\frac{1}{2} R^{-1} \begin{pmatrix} x_0^\top \Phi_A(T, t_0) \sum_{i=1}^m \Phi_i^\top(T, t_0) B_i^\top Q \sum_{i=1}^m \Phi_i(T, t_0) \Phi_A(T, t_0) x_0 \\ \vdots \\ x_0^\top \Phi_A(T, t_0) \sum_{i=1}^m \Phi_i^\top(T, t_0) B_m^\top Q \sum_{i=1}^m \Phi_i(T, t_0) \Phi_A(T, t_0) x_0 \end{pmatrix} \quad (13)$$

where  $\Phi_i(t, t_0)$  is the state transition matrix associated with the matrix  $u_i^* B_i$ .

Proof:

The existence of optimal controls follows from Theorem 2. Then Lemma 2 implies that the optimal solutions satisfy:

$$\begin{aligned}\dot{x}^* &= (A + \sum_{i=1}^m B_i u_i^*) x^* , \quad x(t_0) = x_0 \\ \dot{p}^* &= - (A' + \sum_{i=1}^m B_i' u_i^*) p^* , \quad p(T) = -Qx(T) \\ u^*(t) &= \frac{1}{2} R^{-1} \begin{pmatrix} x^* & B_1' & p^* \\ \vdots & \ddots & \\ x^* & B_m' & p^* \end{pmatrix} .\end{aligned}\tag{14}$$

Let  $R^{-1} = (\gamma_1, \dots, \gamma_m)$  where  $\gamma_i = (\gamma_{il}, \dots, \gamma_{im})'$ , then a direct substitution implies

$$u_i^*(t) = \frac{1}{2} \sum_{j=1}^m \gamma_{ji} (x^* B_j' p^*) \quad i = 1, 2, \dots, m .\tag{15}$$

Differentiate (15) with respect to  $t$  and substitute (14) into it,

$$\frac{d}{dt} u_i^*(t) = \frac{1}{2} \sum_{j=1}^m \gamma_{ji} [\dot{x}^* B_j' p^* + x^* B_j' \dot{p}^*] = 0 , \quad i = 1, 2, \dots, m$$

where the commutivity assumption is strongly used. Therefore,

$$u_i^*(t) = u_i^* = \text{constant}, \quad i = 1, 2, \dots, m.$$

With this constant input function, the solution of (14) can be written as

$$\begin{aligned}
 x^*(t) &= \Phi_A(t, t_0) \prod_{i=1}^m \Phi_i(t, t_0) x_0 \\
 p^*(t) &= \Phi_A'(T, t) \prod_{i=1}^m \Phi_i'(T, t) p(T) \\
 &= -\Phi_A'(T, t) \prod_{i=1}^m \Phi_i'(T, t) Q \Phi_A(T, t_0) \prod_{i=1}^m \Phi_i(T, t_0) x_0
 \end{aligned} \tag{16}$$

from which the desired expression for  $u^*$  can be derived. Q.E.D.

Theorem 3 states the simple character of the minimum energy control problem for commutative bilinear systems, i.e. the optimal controls are simply constant vectors which satisfy the algebraic equation (13). This fact enables us to treat the optimal commutative bilinear system as a fixed linear system for which the explicit solution is immediately available. This result also holds for a slightly more general system in which  $A = A(t)$  is a  $n \times n$  time-varying matrix which commutes with  $B_i$  ( $i=1, \dots, m$ ) because in the proof the time-dependence of  $A$  does not play any role.

In order to obtain  $u^*$  from the nonlinear algebraic equation (13) by iterative schemes, it is interesting to study the uniqueness property of the solution to (13). Next, we will use the property of monotonically increasing maps to show the uniqueness of solutions to (13) for a class of commutative bilinear systems.

#### Definition 2:

A continuous map  $M$  from  $R$  into itself is monotonically increasing if  $x_1 \geq x_2$  implies  $M(x_1) \geq M(x_2)$  for all  $x_1$  and  $x_2$  in  $R$ .

Lemma 3:

Given a monotonically increasing map  $M$ , then the solution to the equation  $x + M(x) = 0$  is unique.

Proof:

The proof is straightforward and left for the reader as an exercise.

Theorem 4:

There is a unique optimal solution to the single-input ( $m=1$ ) commutative bilinear system (7) with cost (9) if the matrix  $B_1'Q + B_1'QB_1$  is nonnegative definite.

Proof:

For the single-input case, Eq. (13) becomes

$$u_1 = -\frac{1}{2} x_0' \Phi_A'(T, t_0) \Phi_1'(T, t_0) B_1' Q \Phi_1(T, t_0) \Phi_A(T, t_0) x_0 \quad (17)$$

Define the map  $M : R \rightarrow R$  by

$$M(u_1) = \frac{1}{2} x_0' \Phi_A'(T, t_0) \Phi_1'(T, t_0) B_1' Q \Phi_1(T, t_0) \Phi_A(T, t_0) x_0$$

then

$$\frac{d}{du_1} M(u_1) = \frac{1}{2} x_0' \Phi_A'(T, t_0) \Phi_1'(T, t_0) [B_1'^2 Q + B_1' Q B_1] \Phi_1(T, t_0) \Phi_A(T, t_0) x_0 \geq 0$$

by hypothesis that  $B_1^T Q + B_1^T Q B_1$  is nonnegative definite. Hence  $M$  is monotonically increasing and Lemma 3 gives the uniqueness of the optimal control  $u_1^*$  to (17). Q.E.D.

The situation turns out to be much more complicated in the multi-input case. One can generalize the concept of a monotonically increasing map into the  $m$ -dimensional Euclidean space as follows.

Definition 3:

A continuous map  $G$  from  $R^m$  into itself is monotonically increasing if  $\langle x_1 - x_2, G(x_1) - G(x_2) \rangle \geq 0$  for all  $x_1$  and  $x_2$  in  $R^m$ , where  $\langle \cdot, \cdot \rangle$  is the usual inner product, i.e.  $\langle x, y \rangle = x^T y$ .

We have a similar lemma as in the scalar case.

Lemma 4:

Given a monotonically increasing map  $G$ , then the solution of the equation  $x + G(x) = 0$ ,  $x \in R^m$  is unique.

Proof:

Again, the proof is left to the reader.

Theorem 5:

There is a unique optimal solution to the commutative bilinear system (7) with cost (9) if the matrix  $R^{-1}Z(v_0)$  is nonnegative definite for all  $v_0$  in  $R^n$ , where  $Z(v_0) = (z_{ij})$  and

$$z_{ij} = v_0' (B_j' B_i Q + B_i' Q B_j) v_0 , \quad i, j = 1, 2, \dots, m. \quad (18)$$

Proof:

The proof is similar to that of Theorem 4 but with some modification. Here, we define  $G(u^*) = \frac{1}{2} R^{-1} g$  where

$$g_j = y_0' \Phi_A'(T, t_0) \prod_{i=1}^m \Phi_i'(T, t_0) B_j' Q \prod_{i=1}^m \Phi_i(T, t_0) \Phi_A(T, t_0) y_0, \quad j = 1, \dots, m$$

and

$$dG(u^*; h) = [\frac{1}{2} R^{-1} z(y_0)] \cdot h \quad h \in \mathbb{R}^m$$

is the Fréchet differential of  $G$  at  $u^*$  with increment  $h$ , and

$$z_{ij} = y_0' \Phi_A'(T, t_0) \prod_{k=1}^m \Phi_k'(T, t_0) [B_j' B_i Q + B_i' Q B_j] \prod_{k=1}^m \Phi_k(T, t_0) \Phi_A(T, t_0) y_0.$$

By hypothesis,  $R^{-1} z(v_0) \geq 0$  which implies that  $G$  is monotonically increasing.<sup>†</sup> Therefore, Lemma 4 gives the uniqueness result.

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<sup>†</sup>Because  $R^{-1} z(v_0) \geq 0$  implies that the Fréchet derivative of  $G$  is monotonically increasing, which implies the monotonicity of  $G$ , see Vainberg [14].

### B. With Terminal Constraint

So far, we have dealt with the minimum energy control problem which includes a terminal cost as defined in (9). Many times the minimum energy control problem is considered with some terminal constraint instead of a quadratic cost term, e.g., one may be only interested in driving a given dynamical system to a certain desired target set at some terminal time (either fixed or free) while the total control energy spent is minimized. In other words, the minimization of control energy is taken over the set  $U_c$  of admissible controls which consists of those input functions which do steer the given system to the desired target set at a certain finite time.

On the other hand, a bilinear system does not generally have the global controllable property as a linear system which has been quite thoroughly characterized. Therefore, we shall limit our attention only to the reachable zone  $Z$  associated with each bilinear system rather than the whole space  $R^n$  as the target set when we are dealing with the cost (10).

#### Definition 4:

A set  $Z(x_0; U)$  is called a reachable set associated with (7) if

$$Z(x_0; U) = \left\{ x(T) \in R^n : x(t_0) = x_0, u(t) \in U, x(t) \text{ satisfies (7) in some} \right. \\ \left. \text{finite time interval } [t_0, T]. \right\}$$

A set  $Z(D; U)$  is called a reachable zone associated with (7) if

$$Z(D; U) = \bigcup_{x_0 \in D} Z(x_0; U).$$

Definition 5:

System (7) is reachable to  $x_1$  with respect to  $x_0$  and  $U$  if there is an input  $u(t) \in U$  which steers it from  $x_0$  to  $x_1$  at some finite time  $T$ . System (7) is constant reachable to  $x_1$  with respect to  $x_0$  if there is a constant input function  $u_c$  which steers it from  $x_0$  to  $x_1$  at some finite time  $T$ .

The following theorem states an interesting property regarding the reachability of commutative bilinear systems.

Theorem 6:

The commutative bilinear system (7) is reachable to  $x_1$  with respect to  $x_0$  and  $L^2([t_0, T], R^m)$  if and only if it is constant reachable to  $x_1$  with respect to  $x_0$ .

Remark:

The Theorem assures that if  $u(t) \in L^2([t_0, T], R^m)$  steers the commutative bilinear system from  $x_0$  to  $x_1$  at  $T$ , then there exists a constant input function  $u_c$  which can do the same job as well. This enables us to study the commutative bilinear system with a class of simple easily implemented input functions, namely, the constant input functions.

Proof:

The sufficiency part is true by definition. To show the necessity part, suppose  $u(t) \in L^2([t_0, T], R^m)$  steers (7) from  $x_0$  to  $x_1$  at some finite time  $T$ . If we can construct a constant input  $u_c$  which also steers (7) from  $x_0$  to  $x_1$  at  $T$ , then we are done.

Since each pair of  $(A, B_i; i=1, \dots, m)$  commute with each other, the solution of (7) can be expressed as:

$$x(t) = \Phi_A(t, t_0) \prod_{i=1}^m \Phi_i(t, t_0) x_0, \text{ where } \Phi_i(t, t_0) = I + \sum_{k=1}^{\infty} \frac{B_i^k}{k!} \int_{t_0}^t u_i(s) ds.$$

$$\text{At } t = T, x(T) = \Phi_A(T, t_0) \prod_{i=1}^m (I + \sum_{k=1}^{\infty} \frac{B_i^k}{k!} \int_{t_0}^T u_i(s) ds) x_0.$$

Choose  $u_c$  such that

$$u_{c_i} = \frac{\int_{t_0}^T u_i(s) ds}{T - t_0} \quad (19)$$

then obviously

$$\Phi_A(T, t_0) \prod_{i=1}^m \Phi_{c_i}(T, t_0) x_0 = \Phi_A(T, t_0) \prod_{i=1}^m \Phi_i(T, t_0) x_0 = x_1. \quad \text{Q.E.D.}$$

Again, this result holds for a slightly more general bilinear system in which  $A = A(t)$  is time-varying provided  $(A(t), B_i; i=1, \dots, m)$  commute with one another. For this class of bilinear systems, the reachable zone is much easier to characterize because one need only consider a constant input  $u_c$  as a set of parameters, then compute the corresponding transition matrices  $\Phi_{c_i}(t, t_0)$ ,  $i = 1, \dots, m$ , which characterize the reachable zone of the given system (7) with a set  $D$  of initial conditions.

Bearing this concept of reachable zone in mind, we may consider the minimum energy control problem (10) of the commutative bilinear system (7) associated with a terminal constraint  $x(T) = x_1$ , some

a priori specified final state. The existence of a constant optimal control is derived via a different approach than that used in Theorem 3. The constant optimal control is determined through the boundary conditions. Then one example is given to show the non-uniqueness of optimal control in general.

Theorem 7:

Given a commutative bilinear system (7) with the cost (10). If  $x_1$  belongs to the reachable set  $Z(x_0; L^2([t_0, T], R^m))$ , then there exists a constant optimal control  $u_c^*$  which steers (7) from  $x_0$  to  $x_1$  at  $T$  and minimizes the associated cost (10). Furthermore,  $u_c^*$  satisfies the nonlinear algebraic equation:

$$x_1 = \Phi_A(T, t_0) \prod_{i=1}^m \Phi_{c_i}(T, t_0) x_0. \quad (20)$$

Proof:

Suppose  $U \subset L^2([t_0, T], R^m)$  is the set of admissible controls, i.e.,  $U = \{u(t) \in L^2([t_0, T], R^m) : \Phi_A(T, t_0) \prod_{i=1}^m \Phi_i(T, t_0) = x_1\} \neq \emptyset$ , by hypothesis. By Theorem 6, there exists a nonempty subset  $E$  consisting of all elements of  $U$  which are constant input functions. For each  $u(t) \in U$ , there is a  $u_c \in E$  such that

$$\int_{t_0}^T u(t) dt = \int_{t_0}^T u_c dt.$$

Comparing the costs associated with  $u(t)$  and  $u_c$ , without loss of generality, let  $R$  be a diagonal matrix with positive elements

$r_i$ ,  $i = 1, \dots, m$ , then

$$\int_{t_0}^T u_c' R u_c dt = \sum_{i=1}^m r_i u_{c_i}^2 (T-t_0) = u_1,$$

$$\int_{t_0}^T u'(t) R u(t) dt = \sum_{i=1}^m r_i \int_{t_0}^T u_i^2(t) dt = u_2.$$

In order to show that  $u_1 \leq u_2$ , it suffices to show that

$$u_{c_i}^2 (T-t_0) \leq \int_{t_0}^T u_i^2(t) dt, \quad i = 1, \dots, m.$$

But this is true because

$$u_{c_i}^2 (T-t_0)^2 = \left( \int_{t_0}^T |u_{c_i}| dt \right)^2 \leq \left( \int_{t_0}^T |u_i(t)| dt \right)^2 \leq \int_{t_0}^T u_i^2(t) dt (T-t_0)$$

$i = 1, \dots, m$ , where the last part is due to Hölder's inequality.

Moreover, the equality is achieved if and only if  $u_i(t) = c$ , a constant. Therefore, if the minimum energy exists, it must be incurred by a constant input  $u_c^* \in E$ . But this is indeed the case because the cost is continuous in  $u_c$ , it attains its minimum in some compact subset of  $E$ .

Since optimal controls are in constant forms, they satisfy (20) by a direct computation of the solution of the 'linear' fixed system (7).

Q.E.D.

Finally, we consider a second order single-input commutative bilinear system as an example to show the non-uniqueness of the optimal control.

Example 1:

Given a bilinear system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} u, \quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x(1) = \begin{pmatrix} 0 \\ -e^1 \end{pmatrix},$$

with the cost to be minimized:

$$J(u) = \int_0^1 u^2(t) dt, \quad u(t) \in L^2([0,1], R).$$

It can be easily seen that  $(A, B_1)$  commute; hence, by Theorem 7 an optimal control  $u_c^*$  satisfies the algebraic equation:

$$\begin{pmatrix} 0 \\ -e^1 \end{pmatrix} = e^1 \begin{pmatrix} \cos u_c^* & \sin u_c^* \\ -\sin u_c^* & \cos u_c^* \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^1 \sin u_c^* \\ e^1 \cos u_c^* \end{pmatrix}.$$

Solving for  $u_c^*$ , we obtain  $u_c^* = k\pi$ ,  $k = \pm 1, \pm 3 \dots$ . In order to have the total control energy minimized, the integers with the smallest absolute value are chosen, i.e.  $k_1 = 1$ ,  $k_2 = -1$ . Clearly,  $u_1^* = \pi$  and  $u_2^* = -\pi$  both incur a minimal cost of  $\pi^2$  and steer

the given system to the desired final state. Hence both  $u_1^*$  and  $u_2^*$  are optimal controls.

The lack of uniqueness of the minimum energy control problem for a commutative system associated with a terminal constraint is, in general, expected because the nonlinear two-point boundary value problem generally does not have a unique solution which prevents the uniqueness of optimal solutions in many cases.

#### IV. Application to a Two-Dimensional Missile Intercept System

It is assumed that for a typical high-speed pursuing missile and short initial range, the maneuvering of the vehicles can be restricted to a two-dimensional plane. Choose the coordinate system fixed in the missile as shown in Figure 1. Denote the angular rate of the missile and the target with respect to a non-rotating reference coordinates as  $u_p$  and  $u_T$ , respectively.

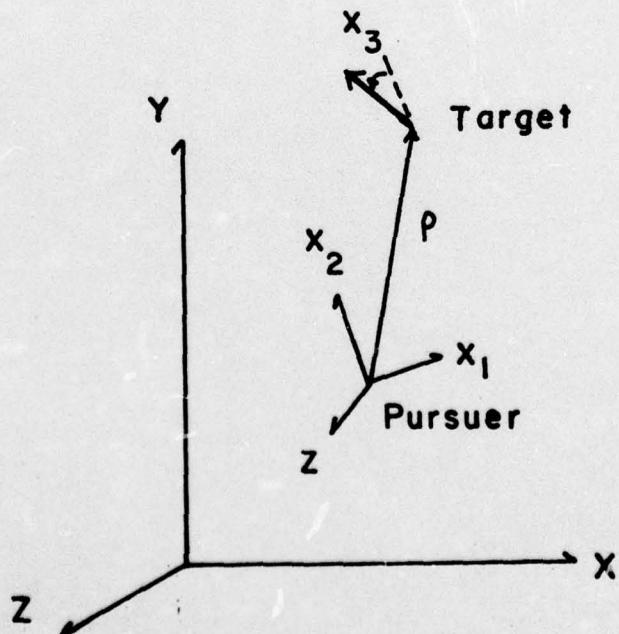


Figure 1.

The equations of motion are then described by [11]

$$\dot{x}_1 = -v_T \sin x_3 + x_2 u_p$$

$$\dot{x}_2 = v_T \cos x_3 - x_1 u_p - v_p \quad (21)$$

$$\dot{x}_3 = u_T - u_p$$

where  $v_T$  and  $v_p$  are the line speeds of the target and the missile relative to air;  $x_1$  and  $x_2$  are the horizontal and vertical distance from the missile, and  $x_3$  is the relative angle between the headings of the missile and target measured counterclockwise.

The system (21) can be transformed into a homogeneous bilinear system by introducing three auxiliary states:  $x_4 = \sin x_3$ ,  $x_5 = \cos x_3$  and  $x_6 = 1$ . That is,

$$\dot{x} = Ax + Bu \quad (22)$$

with

$$A = \begin{pmatrix} 0 & 0 & 0 & -v_T & 0 & 0 \\ 0 & 0 & 0 & 0 & v_T & -v_p \\ 0 & 0 & 0 & 0 & 0 & u_T \\ 0 & 0 & 0 & 0 & u_T & 0 \\ 0 & 0 & 0 & -u_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad x(t_0) = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \\ \sin x_3(t_0) \\ \cos x_3(t_0) \\ 1 \end{pmatrix} \quad (23)$$

in which  $u = u_p$  is defined as the control variable.

The objective is to find a square integrable function  $u^*(t)$  which steers the missile to the target at some finite time  $T$  (a free time formulation) while the total control energy consumed over this interval  $[t_0, T]$  is minimized. The following performance index is thus considered:

$$J(u) = \int_{t_0}^T u^2(t) dt , \quad T > t_0 \quad (24)$$

subject to

$$x_1(T) = x_2(T) = 0 \quad (25)$$

Apart from the classical proportional navigation guidance law, a crucial assumption is proposed relative to the line speed  $v_p$  of the missile which leads to an explicit solution to the afore-mentioned problem. That is,  $v_p$  is taken to be proportional to (an amplification of)  $u_p$  with the proportional parameter  $\gamma$  to be determined by the boundary conditions, provided  $u_p$  is non-zero of course. Hence, we assume

$$v_p = \gamma u_p. \quad (26)$$

With this condition, Equation (23) becomes

$$A = \begin{pmatrix} 0 & 0 & 0 & -v_T & 0 & 0 \\ 0 & 0 & 0 & 0 & v_T & 0 \\ 0 & 0 & 0 & 0 & 0 & u_T \\ 0 & 0 & 0 & 0 & u_T & 0 \\ 0 & 0 & 0 & -u_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (27)$$

Consequently, in the event that  $v_T$  and  $u_T$  are constants, the solution of the system (22) and (27) can be expressed by:

$$\begin{aligned} x_6(t) &= 1, \quad x_5(t) = \cos x_3(t), \quad x_4(t) = \sin x_3(t), \\ x_3(t) &= x_3(t_0) + \int_{t_0}^t [u_T(s) - u(s)] ds \end{aligned} \quad (28)$$

and

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t_0) \cos \int_{t_0}^t u(s) ds + x_2(t_0) \sin \int_{t_0}^t u(s) ds \\ -x_1(t_0) \sin \int_{t_0}^t u(s) ds + x_2(t_0) \cos \int_{t_0}^t u(s) ds \end{pmatrix} \quad (29)$$

$$+ v_T \int_{t_0}^t \begin{pmatrix} -\sin\{x_3(t_0) - \int_{t_0}^\xi u(s) ds + u_T(\xi-t_0)\} \\ \cos\{x_3(t_0) - \int_{t_0}^\xi u(s) ds + u_T(\xi-t_0)\} \end{pmatrix} d\xi + \gamma \begin{pmatrix} \cos \int_{t_0}^t u(s) ds - 1 \\ -\sin \int_{t_0}^t u(s) ds \end{pmatrix}$$

We will first resolve the terminal constraint problem by considering the intercept angle as a parameter, then incorporate the

solutions with the minimum energy problem. Consideration should be given to two separate cases in which  $u_T$  is zero and non-zero.

A. Non-zero Angular Maneuver of the Target ( $u_T \neq 0$ )

From Equation (28), denote the intercept angle by  $\beta$ , i.e.

$$x_3(T) = x_3(t_0) - \int_{t_0}^T u(s) ds + u_T(T-t_0) = \beta. \quad (30)$$

The terminal constraint (25) becomes

$$0 = \begin{pmatrix} [x_1(t_0) + \gamma] \cos\{u_T(T-t_0) + x_3(t_0) - \beta\} + x_2(t_0) \sin\{u_T(T-t_0) + x_3(t_0) - \beta\} - \gamma \\ -[x_1(t_0) + \gamma] \sin\{u_T(T-t_0) + x_3(t_0) - \beta\} + x_2(t_0) \cos\{u_T(T-t_0) + x_3(t_0) - \beta\} \\ + \frac{v_T}{u_T} \begin{pmatrix} \cos \beta - \cos[u_T(T-t_0) - \beta] \\ \sin[u_T(T-t_0) - \beta] + \sin \beta \end{pmatrix} \end{pmatrix} \quad (31)$$

In other words, the terminal constraint problem of the differential system (22) and (27) has been reduced to solving a pair of nonlinear algebraic equations (31) of transcendental type for an appropriate set  $(\gamma, \beta, T)$ . A solution often exists for this case in which the number of unknowns exceeds the number of equations.

The next proposition shows the existence of a triple  $(\gamma, \beta, T)$  which solves (31) for every initial condition  $(x_1(t_0), x_2(t_0), x_3(t_0))$  in  $R^3$ , the analytic expression for this triple, as well as the proof of the proposition, is given in Wei [15].

Proposition 1:

If a constant but non-zero angular maneuver of the target is assumed, then there exists a triple  $(\gamma, \beta, T)$  satisfying (30) and (31) which solves the terminal constraint problem (25) for every  $(x_1(t_0), x_2(t_0), x_3(t_0)) \in \mathbb{R}^3$ . The corresponding proper control action satisfies (30).

B. Zero Angular Maneuver of the Target ( $u_T = 0$ )

In a similar manner, the terminal constraint (27) becomes

$$0 = \begin{cases} [x_1(t_0) + \gamma] \cos[x_3(t_0) - \beta] + x_2(t_0) \sin[x_3(t_0) - \beta] - \gamma - v_T(T-t_0) \sin \beta \\ -[x_1(t_0) + \gamma] \sin[x_3(t_0) - \beta] + x_2(t_0) \cos[x_3(t_0) - \beta] + v_T(T-t_0) \cos \beta \end{cases} \quad (32)$$

where  $\beta$  is as defined in (30) but with  $u_T = 0$ . Again a proper triple  $(\gamma, \beta, T)$  needs to be solved from this equation. The following proposition indicates the region in which every initial condition will introduce a feasible solution for the triple. The proof is given in [15].

Proposition 2:

If a zero angular maneuver of the target is assumed, then there exists a triple  $(\gamma, \beta, T)$  which solves the terminal constraint problem (25) for every  $(x_1(t_0), x_2(t_0), x_3(t_0)) \in \mathbb{R}^3 \setminus E$ , where

$$E = \{(0, y, z) \in \mathbb{R}^3 : \text{either } y > 0, z = (2k+1)\pi, \text{ or } y < 0, z = 2k\pi; k = 0, \pm 1, \dots\}.$$

It can be easily shown that the system (22) and (27) is a commutative bilinear system. Thus, the results in section III are applicable to the minimum energy problem (24) and (25). The set  $U_C$  of admissible controls in this case includes those input functions  $u(t)$  satisfying the algebraic equations (30) and (31) or (32). The next proposition which gives the explicit form of the optimal controls to the problem (24) and (25) is a direct consequence of Theorem 7.

Proposition 3:

Given the system (22) and (27), there exists an optimal control  $u^* \in U_C$  which minimizes the cost (24) and steers the system to  $x_1(T) = x_2(T) = 0$  at some  $T > t_0$  for each appropriate set of initial conditions  $(x_1(t_0), x_2(t_0), x_3(t_0), u_T, v_T)$ . This control is given by

$$u^*(t) = u_T + \frac{x_3(t_0) - \beta}{T - t_0} \quad (33)$$

where  $T$  and  $\beta$  are given as discussed in Propositions 1 and 2.

Proof: Theorem 7 implies the existence of constant optimal controls  $u^* \in U_C$ . By Equation (30),  $u^*$  is given as in (33).

The striking character of this optimal control law in a constant form is not completely without expectation because the control aspects proposed here include two channels. One is the angular maneuver of the missile which counterbalances (offsets) the angular maneuver of

the target, while the additional degree of freedom introduced by  $\gamma$  carries out the major pursuit part of the problem and leads to a simple solution having some intuitive sense.

On the other hand, it should be noted that although an optimal control is in a constant form (a step function), a sub-optimal control law can always be constructed which will drive the missile to the target at some  $T$ , with an appropriate intercept angle  $\beta$  and for some ratio  $\gamma$  as long as the area swept out satisfies (30). This allows the control engineer a great deal of flexibility in the design of a feasible easily implemented sub-optimal controller.

#### V. Conclusion

This paper deals with the bilinear regulator problem from a theoretical point of view. The existence of optimal controls for the quadratic cost problem associated with a bilinear system is established. A class of commutative bilinear systems and the minimum energy problem associated with it are specially investigated. This problem has some interesting features in which the optimal controls are in the form of a constant vector determined by the boundary conditions. On the other hand, as far as the terminal constraint problem is concerned, the system has a variety of different controls to be applied in reaching the desired terminal state provided these controls satisfy an area condition. This has significant meaning from the design point of view. Sufficient conditions for the uniqueness of the minimum energy control are derived for this problem without a terminal constraint. Application to a two-dimensional pursuit-evasion problem is considered in which the analytical expression of

the optimal control is obtained. This illustrates the simplicity in structure of the controllability and minimum energy problem for commutative bilinear systems.

The problems yet to be solved include the construction of a sub-optimal closed-loop control law, the estimation of desired states either in a deterministic or a stochastic sense, and the structure of the optimal controls for different bilinear systems with different performance functions. Another interesting problem which is currently investigated is the singularly perturbed commutative bilinear systems. An example is given in [15] which indicates the possibility to solve the associated optimal control problem analytically. In addition to this, the stability of the optimal bilinear system should also be examined because the constant optimal control enters the system directly in a feedback form with an adjustable gain matrix.

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